GAUSSIAN PROCESSES
EXERCISE SHEET 12: INFINITE DIMENSIONAL MEASURES II

Exercise 1.

Let $r_0 > 0$ be arbitrary. Define dyadic scales

$$r_k := 2^{-k} r_0, \quad k \geq 0.$$

Step 1: Construction of nets. For each k, let $A_k \subset T$ be a minimal r_k-net of T, so $|A_k| \leq N(T, d, r_k)$, and for every $s \in T$ there exists a point $\pi_k(s) \in A_k$ with

$$d(s, \pi_k(s)) < r_k.$$

Let $\pi_k(s)$ denote a nearest net point in A_k to s.

For each scale $k \geq 1$, define the set of net-pairs

$$\mathcal{P}_k := \{(\pi_k(s), \pi_{k-1}(s)) : s \in T\}.$$

Each pair $(a, b) \in \mathcal{P}_k$ represents an increment of the process at scale k:

$$G_a - G_b.$$

For each $k \geq 1$, define the event

$$E_k := \Big\{ \exists (a, b) \in \mathcal{P}_k : |G_a - G_b| > u_k \Big\},$$

where $(u_k)_{k \geq 1}$ are thresholds to be chosen later.

Step 2: Tail bound for each increment. Since G is Gaussian, for each $(a, b) \in \mathcal{P}_k$, the increment $G_a - G_b$ has variance

$$\operatorname{Var}(G_a - G_b) = d(a, b)^2 \leq (2r_{k-1})^2.$$

Hence, for any $u_k > 0$, by the Gaussian tail bound we have

$$\mathbb{P}\big(|G_a - G_b| > u_k\big) \leq 2 \exp\Big(-\frac{u_k^2}{2(2r_{k-1})^2}\Big) = 2 \exp\Big(-\frac{u_k^2}{8r_{k-1}^2}\Big).$$

Let $M_k := |\mathcal{P}_k| \leq |A_k|\,|A_{k-1}|$. Then

$$\mathbb{P}(E_k) \leq \sum_{(a,b) \in \mathcal{P}_k} \mathbb{P}(|G_a - G_b| > u_k) \leq 2M_k \exp\Big(-\frac{u_k^2}{8r_{k-1}^2}\Big).$$

Set

$$u_k := C\, r_k \Big(\sqrt{\log M_k} + k\Big),$$

with a sufficiently large constant $C > 0$. Note that

$$\frac{u_k^2}{8r_{k-1}^2} \geq \frac{C^2 r_k^2 (\log M_k + k^2)}{8r_{k-1}^2} = \frac{C^2}{8} \cdot \frac{r_k^2}{r_{k-1}^2} (\log M_k + k^2).$$

Since $r_k = 2^{-k} r_0$, we have $\frac{r_k^2}{r_{k-1}^2} = 2^{-2}$. Hence

$$\mathbb{P}(E_k) \leq 2M_k \exp\Big(-\frac{C^2}{32}(\log M_k + k^2)\Big) = 2M_k \cdot M_k^{-C^2/32} \cdot e^{-C^2 k^2/32} = 2M_k^{1 - C^2/32} e^{-C^2 k^2/32}.$$

If C is chosen large enough so that $C^2/32 > 2$, then $1 - C^2/32 < -1$, so

$$M_k^{1-C^2/32} \leq 1 \quad ,$$

and the tail $e^{-C^2k^2/32}$ is summable. Therefore,

$$\sum_{k\geq 1} \mathbb{P}(E_k) < \infty.$$

Step 3: Application of Borel–Cantelli lemma. By Borel–Cantelli, with probability one only finitely many E_k occur. Therefore, there exists a full-probability event Ω' such that for all sufficiently large k, for all $(a, b) \in \mathcal{P}_k$,

$$|G_a - G_b| \leq u_k.$$

Step 4: Chaining sum and continuity. For any $t \in T$, we show that

$$G_t = \lim_{k\to\infty} G_{\pi_k(t)}$$

In fact, on Ω', the tail sum $\sum_{k\geq K} |G_{\pi_k(t)} - G_{\pi_{k-1}(t)}| \leq \sum_{k\geq K} u_k$ can be made arbitrarily small by taking K large enough, because

$$\sum_{k\geq K} u_k \lesssim \sum_{k\geq K} r_k \sqrt{\log N(T, d, r_k)} + \sum_{k\geq K} kr_k,$$

and both terms tend to 0 as $K \to \infty$ (entropy integral finite, geometric series convergent). Thus $\lim_{k\to\infty} G_{\pi_k(t)}$ exists with probability one.

Let $\varepsilon > 0$. Then, by Markov's inequality and the definition of the canonical metric,

$$\mathbb{P}\big(|G_s - G_t| > \varepsilon\big) \leq \frac{\mathbb{E}[(G_s - G_t)^2]}{\varepsilon^2} = \frac{d(s,t)^2}{\varepsilon^2}.$$

Hence, as $k \to \infty$ (i.e., $d(\pi_k(t), t) \to 0$),

$$\mathbb{P}\big(|G_{\pi_k(t)} - G_t| > \varepsilon\big) \longrightarrow 0,$$

which shows that $G_{\pi_k(t)} \to G_t$ *in probability* for any $t \in T$.

From this convergence in probability, one can extract a deterministic subsequence k_n such that

$$G_{\pi_{k_n}(t)} \to G_t \quad \text{almost surely.}$$

Note that $\lim_{k\to\infty} G_{\pi_k(t)}$ exists, we have that $G_t = \lim_{k\to\infty} G_{\pi_k(t)}$.

Fix t. For any s closed enough to t, we can replace the first few $\pi_k(s)$ by $\pi_k(t)$ in the limit $G_s = \lim_{k\to\infty} G_{\pi_k(s)}$ if $d(s, \pi_k(t)) < r_k$, note that the remainder terms $\sum_{k\geq K} |G_{\pi_k(t)} - G_{\pi_{k-1}(t)}|$ and $\sum_{k\geq K} |G_{\pi_k(s)} - G_{\pi_{k-1}(s)}|$ both tend to 0 on Ω' by the computation above, we already get continuity at t almost surely.

Step 5: Continuity on all of T. Since T is countable, intersect the full-probability events for each $t \in T$ to obtain a single probability-one event on which G is continuous at every $t \in T$.

Remark. One may check the Kolmogorov continuity theorem and compare this proof with it if interested. Do you think this proof still works if T is uncountable? □

Exercise 2.

(a) A_k is a Hilbert space.

Linearity, symmetry and positive-definiteness are immediate. For completeness, define the map $T_k : A_k \to \ell^2$ by

$$(T_k \bar{a})_n := n^{\frac{k}{2}} a_n.$$

Then

$$\|T_k\bar{a}\|_{\ell^2}^2 = \sum_{n\geq 1}(n^{\frac{k}{2}}a_n)^2 = \sum_{n\geq 1} a_n^2 n^k = \|\bar{a}\|_k^2,$$

so T_k is an isometry. In particular,

$$\langle \bar{a}, \bar{b}\rangle_k = \langle T_k\bar{a}, T_k\bar{b}\rangle_{\ell^2}.$$

If $(\bar{a}^{(m)})$ is Cauchy in $\|\cdot\|_k$, then $(T_k\bar{a}^{(m)})$ is Cauchy in ℓ^2, which is complete. Thus $T_k\bar{a}^{(m)} \to x \in \ell^2$. Writing $a_n := n^{-\frac{k}{2}}x_n$ gives $\bar{a} \in A_k$ and $T_k\bar{a} = x$, so $\bar{a}^{(m)} \to \bar{a}$ in $\|\cdot\|_k$. Hence A_k is complete and therefore a Hilbert space.

(b) (c) You may check Section 4.5.2 in the notes. $\square$